A Note on the Utility Function Under Prospect Theory

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Abstract

We show that preference-homogeneity and loss-aversion are necessary and sufficient for the value function to have the power form with identical powers for gains and losses and for the probability weighting functions for gains and losses to be identical.

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1. Introduction

It is well known that the axioms of expected utility are violated in a range of experiments and surveys. This has been well known for a long time; for instance, Luce and Raiffa (1957).[1] For a more recent and definitive treatment, see Kahneman and Tversky (2000).

The main behavioral alternative to expected utility is prospect theory. The earliest version was given by Kahneman and Tversky (1979). A later version based on cumulative transformations of probability and, hence, the insights developed in rank dependent expected utility,[2] was provided by Tversky and Kahneman (1992).

Prospect theory has proven extremely influential in explaining a range of phenomena that could not be otherwise explained within an expected utility framework. These include the disposition effect, asymmetric price elasticities, elasticities of labour supply that are inconsistent with standard models of labour supply and the excess sensitivity of consumption to income; see, for example, Camerer (2000). Further applications include the explanation of tax evasion (Dhami and al-Nowaihi (2007)) and several applications to finance (Thaler (2005)) among others.

A critical aspect in successfully applying prospect theory, particularly in quantitative applications, is the form of the utility function for gains and losses. Tversky and Kahneman (1992) state, without proof, that if preference homogeneity[3] holds, then the value function of prospect theory has the power function form[4]

$$v(x) = x^{\alpha}, \text{ for } x \geq 0,\ v(x) = -\lambda(-x)^{\beta}, \text{ for } x < 0, \text{ where } \alpha > 0,\ \beta > 0,\ \lambda > 0. \quad (1.1)$$

with loss aversion implying that $\lambda > 1$.

The contribution of our paper is as follows. We give a simple proof which shows that preference homogeneity is a necessary and sufficient condition for the preferences given in (1.1). Furthermore, loss aversion implies that, not only $\lambda > 1$, but also $\alpha = \beta$. Finally, we show that the probability weighting function for losses must be the same as that for gains. These results are in agreement with the empirical evidence (Tversky and Kahneman (1992) and Prelec (1998)).

Section 2 gives the basic definitions that we need for our main theorem, which is derived in Section 3. Section 4 concludes.

[1] Luce and Raiffa (1957, p35) wrote "A second difficulty in attempting to ascertain a utility function is the fact that reported preferences almost never satisfy the axioms..."

[2] This mainly had to do with a transformation of cumulative rather than objective probabilities; see Quiggin (1993) for the details.

[3] Preference homogeneity is formally defined below. It essentially implies that when all prizes in a lottery are scaled up by a factor, say k, then the certainty equivalent of the lottery is also scaled up by the same factor k.

[4] Under expected utility theory, preference homogeneity gives rise to CRRA preferences.

2. Preliminary definitions

We shall use the following notation: $(x;p)$ stands for the *simple lottery* that pays $x \in \mathbf{R}$ with probability $p \in [0,1]$ and 0 otherwise. $(\mathbf{x};\mathbf{p})$, given by

$$(\mathbf{x};\mathbf{p}) = (x_{-m}, x_{-m+1}, ..., x_{-1}, x_0, x_1, x_2, ..., x_n;\ p_{-m}, p_{-m+1}, ..., p_{-1}, p_0,\ p_1, p_2, ..., p_n),$$

stands for the lottery that pays $x_i \in \mathbf{R}$ with probability $p_i \in [0,1]$, where $\sum_{i=-m}^{n} p_i = 1$ and $x_{-m} \leq x_{-m+1} \leq ... \leq x_{-1} \leq x_0 = 0 \leq x_1 \leq x_2 \leq ... \leq x_n$. If $\mathbf{x} = (x_1, x_2, ..., x_n)$ then $-\mathbf{x} = (-x_1, -x_2, ..., -x_n)$ and $\mathbf{x}^r = (x_n, x_{n-1}, ..., x_1)$. Thus $(-\mathbf{x}^r; \mathbf{p}^r)$ stands for the lottery that pays $-x_i$ with probability p_i. If each $x_i \geq 0$ and, for some i, $p_i x_i > 0$, then we call $(\mathbf{x};\mathbf{p})$ a *positive lottery*.

Definition 1 *: (Tversky and Kahneman, 1992) The decision maker exhibits preference homogeneity if, for all lotteries,* $(\mathbf{x};\mathbf{p})$*, if* c *is the certainty equivalent of* $(\mathbf{x};\mathbf{p})$ *then, for all* $k \in \mathbf{R}_+$*,* kc *is the certainty equivalent of* $(k\mathbf{x};\mathbf{p})$.

Definition 2 *: (Kahneman and Tversky, 1979)* $v : \mathbf{R} \to \mathbf{R}$ *is a value function over riskless outcomes, if* $v(0) = 0$ *(reference dependence) and* v *is strictly increasing (monotonicity). Furthermore, if* $|v(-x)| > v(x)$ *for* $x > 0$ *then* v *exhibits loss aversion.*[5]

Definition 3 *: By a probability weighting function we mean a strictly increasing function* $w : [0,1] \overset{onto}{\to} [0,1], w(0) = 0, w(1) = 1.$

Definition 4 *: (Tversky and Kahneman, 1992) Let the probability weighting function for gains be* w_+ *and let the probability weighting function for losses be* w_-*. For cumulative prospect theory, the decision weights,* π_i*, are defined as follows:*

$\pi_n = w_+(p_n)$,
$\pi_{n-1} = w_+(p_{n-1} + p_n) - w_+(p_n)$,
...
$\pi_i = w_+\left(\Sigma_{j=i}^{n}\ p_j\right) - w_+\left(\Sigma_{j=i+1}^{n}\ p_j\right)$,
...
$\pi_1 = w_+\left(\Sigma_{j=1}^{n}\ p_j\right) - w_+\left(\Sigma_{j=2}^{n}\ p_j\right)$,
$\pi_0 = w_+\left(\Sigma_{j=0}^{n}\ p_j\right) - w_+\left(\Sigma_{j=1}^{n}\ p_j\right)$,
$\pi_{-m} = w_-(p_{-m})$,
$\pi_{-m+1} = w_-(p_{-m} + p_{-m+1}) - w_-(p_{-m})$,
...

[5]It is usual to impose the further restrictions: v is continuous, v is concave for $x \geq 0$ (declining sensitivity for gains) and v is convex for $x < 0$ (declining sensitivity for losses). However, these extra assumptions will play no part in this paper.

$\pi_j = w_- \left(\Sigma_{i=-m}^{j}\, p_i\right) - w_- \left(\Sigma_{i=-m}^{j-1}\, p_i\right),$

...

$\pi_{-1} = w_- \left(\Sigma_{i=-m}^{-1}\, p_i\right) - w_- \left(\Sigma_{i=-m}^{-2}\, p_i\right).$

The value of the lottery to the decision maker is given by $V(\mathbf{x};\mathbf{p}) = \Sigma_{i=-m}^{n}\, \pi_i v(x_i)$.

Definition 5 *: Loss aversion holds for positive lotteries if, for some* $\lambda > 1$, $\frac{|V(-\mathbf{x}^r;\, \mathbf{p}^r)|}{V(\mathbf{x};\, \mathbf{p})} = \lambda$, *for all positive lotteries. We call* λ *the coefficient of loss aversion.*[6]

3. Derivation of the power form for the value function

We derive our main results in this section: Preference homogeneity for simple lotteries is sufficient for the value function for riskless outcomes to have the power form. It then follows that preference homogeneity must hold for all lotteries. If we add loss aversion for riskless outcomes, then the power for losses (β in (1.1)) must be the same as that for gains (α in (1.1)). Furthermore, the coefficient of loss aversion (λ in (1.1)) must be greater than 1. If we extend loss aversion to apply to simple lotteries as well, then the probability weighting functions for losses and gains must be identical. It then follows that loss aversion must hold for all positive lotteries. Theorem 1, below, formalizes these results.

Theorem 1 *: (a) If preference homogeneity holds for simple lotteries, then the value function for riskless outcomes,* v*, takes the form:*

$$v(x) = x^{\alpha},\ \text{for } x \geq 0,\ v(x) = -\lambda(-x)^{\beta},\ \text{for } x < 0,\ \text{where } \alpha > 0,\ \beta > 0,\ \lambda > 0. \tag{3.1}$$

Conversely, if the value function for riskless outcomes takes the form (3.1), then preference homogeneity holds for all lotteries.

(b) If the value function (3.1) for riskless outcomes exhibits loss aversion, then $\lambda > 1$ *and* $\alpha = \beta$.

(c) If preference homogeneity and loss aversion both hold for simple lotteries, then the value function for riskless outcomes takes the form (3.1) with $\lambda > 1$, $\alpha = \beta$ *and* $w_- = w_+$. *Conversely, if the value function for riskless outcomes takes the form (3.1) with* $\lambda > 1$, $\alpha = \beta$ *and* $w_- = w_+$*, then preference homogeneity holds for all lotteries and loss aversion holds for all positive lotteries.*[7]

[6] Tversky and Kahneman (1992) define loss aversion only for riskless outcomes. Definition 5 is an attempt to extend this concept to positive lotteries. The following example shows it cannot be extended to lotteries with both losses and gains. Consider the value function $v(x) = x$, $x \geq 0$; $v(x) = 2x$, $x < 0$ and the probability weighting function $w_-(p) = w_+(p) = p$. For riskless outcomes and simple lotteries, $\lambda = 2$. However, for $(\mathbf{x};\mathbf{p}) = (-2, 0, 6; 0.5, 0, 0.5)$, we have $(-\mathbf{x}^r;\mathbf{p}^r) = (-6, 0, 2; 0.5, 0, 0.5)$ and, hence, $\lambda = \frac{|V(-\mathbf{x}^r;\, \mathbf{p}^r)|}{V(\mathbf{x};\, \mathbf{p})} = \frac{5}{1} = 5 \neq 2$.

[7] Part (c) was added in response to the insightful comments of the referee.

Proof: (a) Let $0 \leq c \leq 1$. By reference dependence and monotonicity, $0 = v(0) \leq v(c) \leq v(1)$ and $v(1) > 0$. Hence, $0 \leq \frac{v(c)}{v(1)} \leq 1$. Let $p = w_+^{-1}\left(\frac{v(c)}{v(1)}\right)$. Hence, $w_+(p) = \frac{v(c)}{v(1)}$ and, hence,

$$w_+(p)\,v(1) = v(c)\,. \tag{3.2}$$

Hence, c is the certainty equivalent of $(1; p)$. Preference homogeneity for simple lotteries then implies,

$$w_+(p)\,v(k) = v(ck)\,,\text{ for all } k \geq 0. \tag{3.3}$$

Substitute $w_+(p) = \frac{v(c)}{v(1)}$ from (3.2) into (3.3) to get

$$v(ck) = \frac{v(c)\,v(k)}{v(1)},\text{ for all } c \in [0,1] \text{ and all } k \geq 0. \tag{3.4}$$

Define $u : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$u(x) = \frac{v(x)}{v(1)},\ x \geq 0. \tag{3.5}$$

In particular, for $x = 1$, (3.5) gives

$$u(1) = 1. \tag{3.6}$$

From (3.4) and (3.5) we get $u(ck) = \frac{v(ck)}{v(1)} = \frac{v(c)v(k)}{v(1)v(1)} = u(c)\,u(k)$. Hence,

$$u(ck) = u(c)\,u(k)\,,\text{ for all } c \in [0,1] \text{ and all } k \geq 0. \tag{3.7}$$

Equation (3.7) holds for any numbers c, k such that $c \in [0,1]$ and all $k \geq 0$. In what follows, c does not necessarily have the interpretation of a certainty equivalent.

Let $x > 0$. If $0 < x \leq 1$, let $c = x$ and $k = \frac{1}{x}$. If $x > 1$, let $c = \frac{1}{x}$ and $k = x$. In either case, (3.6) and (3.7) give $1 = u(1) = u\left(x\frac{1}{x}\right) = u\left(\frac{1}{x}\right)u(x)$. Hence,

$$u\left(\frac{1}{x}\right) = \frac{1}{u(x)},\text{ for all } x > 0. \tag{3.8}$$

Let $x \geq 0$ and $y \geq 0$. If $x \leq 1$, take $c = x$ and $k = y$. If $y \leq 1$, take $c = y$ and $k = x$. In either case, (3.7) gives $u(xy) = u(x)\,u(y)$. Suppose now $x > 1$ and $y > 1$. Then (3.7) and (3.8) give $u(xy) = \frac{1}{u\left(\frac{1}{xy}\right)} = \frac{1}{u\left(\frac{1}{x}\right)}\frac{1}{u\left(\frac{1}{y}\right)} = u(x)\,u(y)$. Hence,

$$u(xy) = u(x)\,u(y) \text{ for all } x \geq 0 \text{ and all } y \geq 0 \tag{3.9}$$

Since v is strictly increasing, so u is also strictly increasing (from (3.5), since $v(1) > 0$). Hence, (3.9) has the unique solution[8]:

$$u(x) = x^{\alpha},\text{ for some } \alpha > 0. \tag{3.10}$$

[8] See, for example, Eichhorn (1978, Theorem 1.9.13).

Putting $a = v(1)$, (3.5) and (3.10) give:

$$v(x) = ax^{\alpha},\ a > 0,\ \alpha > 0,\ x \geq 0 \tag{3.11}$$

Similarly, by now taking $u(x) = \frac{v(-x)}{v(-1)}$, $x \geq 0$, and using the probability, w_- for losses, we get

$$v(x) = -\lambda(-x)^{\beta},\ b > 0,\ \lambda > 0,\ x \leq 0 \tag{3.12}$$

Without loss of generality, we can take $a = 1$, so that

$$v(x) = x^{\alpha},\text{ for } x \geq 0,\ v(x) = -\lambda(-x)^{\beta},\text{ for } x < 0,\text{ where } \alpha > 0,\ \beta > 0,\ \lambda > 0. \tag{3.13}$$

(b) Loss aversion then implies

$$\lambda x^{\beta} > x^{\alpha} \text{ for all } x > 0. \tag{3.14}$$

For $x = 1$, (3.14) gives

$$\lambda > 1 \tag{3.15}$$

Also from (3.14)

$$\ln\lambda > (\alpha - \beta)\ln x \text{ for all } x > 0. \tag{3.16}$$

We will now prove that $\alpha = \beta$. Suppose $\alpha \neq \beta$. Then either $\alpha > \beta$ or $\beta > \alpha$. If $\alpha > \beta$, then we can make $(\alpha - \beta)\ln x$ as large as we like by choosing x to be sufficiently large. But this cannot be because, by (3.16), $(\alpha - \beta)\ln x$ is bounded above by $\ln\lambda$. If $\beta > \alpha$, then we can make $(\alpha - \beta)\ln x$ as large as we like by choosing $x > 0$ sufficiently close to 0. But this cannot be true either. Hence $\alpha = \beta$.

(c) Let $0 \leq p \leq 1$. Then $V(1;p) = w_+(p)v(1)$ and $V(-1;p) = \lambda w_-(p)v(-1)$. By loss aversion (Definition 5), $|V(-1;p)| = \lambda V(1;p)$. Hence $\lambda w_-(p) = \lambda w_+(p)$. Hence $w_-(p) = w_+(p)$.

Simple calculations, using Definitions 1 to 5, show that the converses of the above statements hold.

4. Conclusions

We provide a formal proof which shows that preference homogeneity is necessary and sufficient for the power function form of the utility function proposed in Tversky and Kahneman (1992). We also show that loss aversion gives rise to a more parsimonious utility function than that proposed by Tversky and Kahneman (1992). We also show that the probability weighting function for losses must be the same as that for gains. These results are consistent with the empirical evidence and, by reducing the number of free parameters, they are expected to simplify the application of prospect theory. In applications such reduction in free parameters is appealed to on grounds of convenience. We provide, instead, a rigorous theoretical justification.

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