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Asset Return and Wealth Dynamics with Reference Dependent Preferences and Heterogeneous Beliefs*

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Abstract. We study a model of a financial market populated with heterogenous agents whose preferences exhibit dependence on some reference level of wealth. Investment decisions of the agents are myopic and are based upon the demand for the risky asset derived from an S-shaped utility maximization. The specific demand form allows to model both heterogeneity of the system relative to the reference points of the agents and heterogeneity with respect to their beliefs about the future asset return. We analyze the impact of the former layer of heterogeneity on the asset return and wealth dynamics.

Keywords: demand for the risky asset, S-shaped utility, reference point, heterogeneous beliefs, asset return and wealth dynamics.

JEL classification: C62, C63, D84, G12.

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1. Introduction

Financial markets modelling is a constantly growing research area within the field of financial economics. Artificial financial markets are typically populated with a large number of various traders, with the impact of a single market participant on the asset price dynamics being negligibly small. Each trader, however, may influence the behavior of a small group of other traders who, in turn, can influence the whole market to some extent. Financial markets, thus, are viewed as a "soup" of diverse agents who interact with each other making the market resemble a constantly boiling mixture.

Multi-agent modelling is a major tool to cope with such dramatically complex systems. It allows to employ the bottom-up approach, focusing on the micro level of the agents interaction but aiming at studying the macro effects of the asset price dynamics. Models of this kind are relatively simple and often analytically tractable. Due to the mostly analytical nature of multi-agent models, closed-form solutions can often be found, otherwise advanced numerical methods of nonlinear dynamics, bifurcation theory, or computer simulations can be applied for the analysis.

The initial steps on the way to the multi-agent approach were taken by several scholars in the early nineties, among which are Day and Huang (1990), DeLong et al. (1990), Chiarella (1992), Kirman (1993) and Lux (1995). These early models focused mostly on the stylized analysis of simple behavioral rules as a cause of endogenous price fluctuations. More computationally oriented models evolved later from the first attempts to create real-proportion artificial stock markets simulated on computers. The Santa Fe market (Arthur et al., 1997) is an example of such an approach. Some of the subsequent computational models are Großklags et al. (2000), Chen and Yeh (2001), Chen et al. (2001) and Duffy (2001). Two excellent recent surveys of the state of the art in the field are Hommes (2006) and LeBaron (2006).

As a rule, multi-agent modelling assumes that market traders are not identical but differ with respect to their expectations (or beliefs) about the future asset price. Such models distinguish themselves from representative agent models that prevailed in economic research for a long period of time. This informational source of heterogeneity provokes trade in the market between the agents since their beliefs affect and change their demand for the assets.

There may be another source of heterogeneity, which stems from the behavioral aspects of the agents' actions and is intimately connected with their preferences. This second type of heterogeneity can arise from a specific form of the demand for the risky assets as long as it incorporates reference dependence. Heterogeneity with respect to reference point has not been studied before and the aim of this paper is to analyze how it affects asset price or return and wealth dynamics.

A prominent approach which utilizes reference dependence is the prospect theory of Kahneman and Tversky (1979), which has been a cornerstone of research of decision making under risk and uncertainty in behavioral economics and psychology. Its descriptive properties, namely the existence of some reference point with respect to which the agents code outcomes into gains and losses, allow to model behavior of economic agents in a more realistic way than the mainstream notion of the expected utility does. Reference dependence arises from the special S-shaped form of the utility function in contrast to the everywhere concave function of the expected utility theory. This feature appears to be also the source of the reflection effect (opposite preferences for positive and negative outcomes), a well-documented bias of human decision making.

The model described in this work is partially based on the models of Brock and Hommes (1998) and Chiarella and He (2001). In the former, several types of agents have diverse beliefs about the future, which they adapt using the past history by switching from one strategy to another according to a given fitness measure. With the help of nonlinear dynamics methods the authors showed that a market populated with such heterogeneous agents trading repeatedly in a Walrasian framework combined with evolutionary updating of their beliefs is able to replicate some of the stylized facts and reproduce asset price chaotic behavior. Anufriev and Bottazzi (2004) supplemented this model with heterogeneous horizons of the agents, while Anufriev and Panchenko (2006) investigated the changes in the model outcomes when different market mechanisms are introduced. de Fontnouvelle (2000) enriched it with various information flow schemes about the dividend payments, while Brock et al. (2005) examined an extension to many trader types. Gaunersdorfer (2000) introduced heterogeneity of beliefs with respect to the returns volatility, Hommes et al. (2005) included a market maker into the market pricing mechanism, Brock et al. (2006) introduced risk hedging instruments in form of Arrow securities in the model, and Panchenko et al. (2007) analyzed how local interactions affect the model outcomes.

The work of Chiarella and He (2001) is, in essence, a generalization of Brock and Hommes (1998). The endogenous switching between the trading strategies is, however, not implemented because of complications due to the incorporated wealth dynamics. While the model of Brock and Hommes (1998) can dispense with the dynamics of the agents' wealth as a result of the constant absolute risk aversion (CARA) property of the utility functions, the work of Chiarella and He (2001) deals with it explicitly, making the model a great deal more challenging to handle. The authors first present a growth model of the asset price and wealth, and then they derive a stationary model with respect to the asset return and wealth proportions in order to retain analytical tractability.

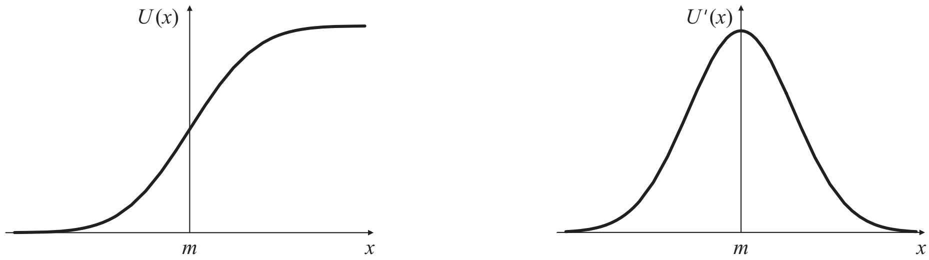
We incorporate reference dependence into a modification of the models of Chiarella and He (2001) and Brock and Hommes (1998) and analyze how it affects the market dynamics by considering two distinct reference points. We study the behavior of the resulting dynamical systems by means of numerical analysis and computer simulations.

The structure of the paper is as follows. In the next section reference dependent preferences are discussed, the demand for the risky asset is constructed, and the beliefs of the agents are introduced. Section 3 focuses on the special case of the reference point, which is taken equal to the current wealth of the agents. Section 4 describes the model with the reference point set equal to the risk-free investment. Section 5 concludes the paper.

2. Heterogeneous belief model with reference dependence

In this section we derive a general model without specific reference points of the agents. We will present two special cases of reference points, analyze their effect on the final system, and dwell on the computer simulation results in the following sections.

The reference dependent feature of the model shows up in light of the specific utility function assumption. In fact, we assume that the utility function $U(x)$ is of an S-shaped form, that is, the function is strictly increasing, continuous, twice differentiable, bounded, and symmetric around its unique inflection point m, which serves as the reference point (see Figure 1). It follows immediately that the agents are strictly risk averse over the outcomes greater than m, and they are strictly risk seeking over the outcomes less than m. At the reference point the agents are locally risk neutral.



(a) An S-shaped function (b) The first derivative

Figure 1: An S-shaped function and its first derivative.

Suppose now that a myopic agent i with the initial wealth $W_{i,t} > 0$ maximizes the expected value of her S-shaped utility of wealth $U(\boldsymbol{W}_{i,t+1})$ by choosing every period of time a proportion of her wealth $\pi_{i,t}W_t$ to be invested in the risky asset, allocating the rest of her wealth $(1-\pi_{i,t})W_t$ to the risk-free one.[1] The risky asset pays an uncertain return $\boldsymbol{R}_{t+1}$, while the risk-free asset is perfectly elastically supplied and pays a constant gross return $R_f = 1 + r_f$. The ex post income of the agent at date $t+1$ after the risky asset rate of return is realized is given by

$$\boldsymbol{W}_{i,t+1} = \pi_{i,t}W_{i,t}\boldsymbol{R}_{t+1} + (1-\pi_{i,t})W_{i,t}R_f = W_{i,t}R_f + \pi_{i,t}W_{i,t}(\boldsymbol{R}_{t+1} - R_f). \quad (1)$$

The model does not include consumption as the wealth of the agents is entirely reinvested in assets every period of time.

The demand for the risky asset $\pi_{i,t} = \text{argmax}_{\pi_{i,t}}\{U(\boldsymbol{W}_{i,t+1})\}$ has been derived in a closed form for a special case of a general S-shaped utility, namely, for the normal utility function in Gerasymchuk (2007). The demand is given by

$$\pi_{i,t} = \frac{s_i^2(\mathrm{E}_{i,t}[\boldsymbol{R}_{t+1}] - R_f)}{W_{i,t}(W_{i,t}R_f - m_{i,t})\mathrm{Var}_{i,t}[\boldsymbol{R}_{t+1}]}, \quad (2)$$

where $m_{i,t}$ is the location parameter of the utility function, i.e. the reference point of the agent, and s_i is the scale parameter of $U(\boldsymbol{W}_{i,t+1})$, which accounts for the curvature of the utility. Conditional expectation and conditional variance $\mathrm{E}_{i,t}$ and $\mathrm{Var}_{i,t}$ are based on the publicly available information set $I_t = \{R_{t-1}, R_{t-2}, ...\}$, thus representing the predictors

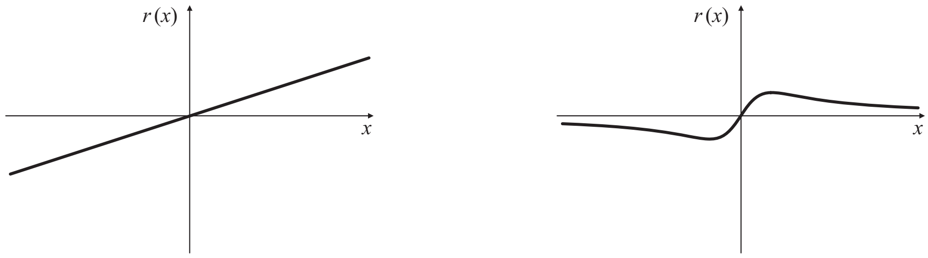
[1] Bold face type denotes random variables at time $t+1$.

(or beliefs) of the agent i about, respectively, the mean and the variance of the risky asset return $\boldsymbol{R}_{t+1}$.[2]

Gerasymchuk (2007) explores the coefficient of absolute risk aversion of the normal utility and argues that this utility does not portray realistic risk attitude of humans in full as the risk aversion function is increasing, although it approximates the behavior quite well for relatively small outcomes (Figure 2a). The risk aversion function is the following:

$$r(W_{i,t}) = \frac{W_{i,t}R_f - m_{i,t}}{s_i^2}. \quad (3)$$

Four different alternatives to the normal utility are elaborated and presented, namely the arctangent, the exponential arctangent, the hyperbolic tangent, and the logistic function, that all exhibit more appealing risk attitude.[3] In fact, the arctangent's risk aversion function appears to be the most realistic one (Figure 2b), although in contrast to the normal utility, there is no close form demand for the risky asset available for this utility.



(a) Normal utility

(b) Arctangent function

Figure 2: Risk aversion function for normal utility and U(x)=arctan(x).

In this paper we take a side-step from a fully microeconomically founded multi-agent model in favor of a more realistic risk aversion function and employ the arctangent as an S-shaped utility:

$$U(W_{i,t}) = \arctan(s_i[W_{i,t} - m_{i,t}]), \quad (4)$$

where, as for the normal utility, $m_{i,t}$ is the location parameter or the reference point of the agent i, and s_i is the scale parameter of the utility function responsible for its curvature.

We, thus, do not derive the demand for the risky asset analytically via the maximization of the underlying utility. We construct it on the basis of the available demands for the normal and exponential (CARA) utilities taking into account the expression for the risk aversion function of the arctangent, which is given by

$$r(W_{i,t}) = \frac{2s_i^2(W_{i,t} - m_{i,t})}{1 + (s_i[W_{i,t} - m_{i,t}])^2}. \quad (5)$$

[2]The return R_t is usually not included in the information set I_t, especially whenever the Walrasian scenario is used for the market clearing price derivation, in order to retain explicit form of the final system.

[3]Section 3 in Gerasymchuk (2007).

In particular, we notice that the demand for both the normal and the CARA utilities is proportional to the Sharpe ratio with the coefficient of proportionality being the inverse of the risk aversion:

$$\alpha_{i,t}^{normal} = \frac{s_i^2}{W_{i,t}R_f - m_{i,t}} \cdot \frac{\mathrm{E}_{i,t}[\boldsymbol{R}_{t+1}] - R_f}{\mathrm{Var}_{i,t}[\boldsymbol{R}_{t+1}]}, \tag{6}$$

$$\alpha_{i,t}^{CARA} = \frac{1}{r} \cdot \frac{\mathrm{E}_{i,t}[\boldsymbol{R}_{t+1}] - R_f}{\mathrm{Var}_{i,t}[\boldsymbol{R}_{t+1}]}, \tag{7}$$

where r is the risk aversion coefficient of the CARA utility, and $\alpha_{i,t}$ is the amount to be invested in the risky asset.

The relationship between the demand for the risky asset, the risk aversion coefficient, and the Sharpe ratio is intuitively clear. The inverse proportionality of the demand to the risk aversion seems quite natural as an investor would invest less money in the risky asset whenever her risk aversion is large. The Sharpe ratio, on the other hand, expresses humans' inherent desire for maximizing possible gains while minimizing possible losses: an agent would buy more shares of the risky asset when she perceives a high next period excess return, and she would buy less shares otherwise.

It is difficult to say whether this particular form of the demand (the Sharpe ratio multiplied by the inverse of the risk aversion coefficient) holds in general, but some of its qualitative properties should certainly hold for a plausible description of financial decision making. In particular, the demand should be in the form of a gain-loss functional increasing in gains and decreasing in losses, the Sharpe ratio being a special case of such given that gains are associated with the expected value of the return and risk is attributed to its variance. Also the inverse relation with the risk aversion coefficient should hold for any reasonable demand function, otherwise an investor may exhibit actions inconsistent with empirical evidence.

Analogously to (6) and (7), we construct the demand for the arctangent utility by multiplying the inverse of its risk aversion (5) by the Sharpe ratio:

$$\alpha_{i,t}^{arctan} = \frac{1 + (s_i[W_{i,t} - m_{i,t}])^2}{2s_i^2(W_{i,t} - m_{i,t})} \cdot \frac{\mathrm{E}_{i,t}[\boldsymbol{R}_{t+1}] - R_f}{\mathrm{Var}_{i,t}[\boldsymbol{R}_{t+1}]}. \tag{8}$$

For the ease of the further results derivation we set $s_i = 1$ and we also assume that the agents hold heterogenous beliefs in the sense of different conditional expectations but equal and constant conditional variances $\mathrm{Var}_{i,t}[\boldsymbol{R}_{t+1}] = 1$ at any period of time. Gaunersdorfer (2000) considered the model of Brock and Hommes (1998) with variances changing over time and obtained similar results as in the case of constant ones.

Consequently, taking into account that $\alpha_{i,t} = \pi_{i,t} W_{i,t}$, the optimal proportion of wealth to be invested in the risky asset relevant to the arctangent utility function becomes

$$\pi_{i,t} = \frac{(1 + [W_{i,t} - m_{i,t}]^2)\ \mathrm{E}_{i,t}[\boldsymbol{R}_{t+1} - R_f]}{2W_{i,t}(W_{i,t} - m_{i,t})}. \tag{9}$$

Everything up to this point was formulated with respect to a single agent. Now we turn to the analysis of groups of agents assuming that the agents are not being able to migrate from one group to another. We suppose that there are two sources of heterogeneity in the market (heterogeneity of beliefs and heterogeneity of reference points) and discuss both below in detail.

2.1. Heterogenous beliefs

Concerning the first source of heterogeneity with respect to the beliefs of the agents $\mathrm{E}_{j,t}[\boldsymbol{R}_{t+1} - R_f]$, where j stands for a particular belief form, we assume the following. There are two types of agents in the market: *fundamentalists* and *chartists*, who are chosen to represent in a stylized way to distinct strategies usually employed in the real financial markets. The agents hold identical beliefs about the future asset return within each type.

Fundamentalists believe that the future price of the asset will be equal to its fundamental value, that is, they forecast the excess conditional mean of the future return (from the risk-free rate) equal to the risk premium δ:

$$\mathrm{E}_{f,t}[\boldsymbol{R}_{t+1} - R_f] = \delta. \tag{10}$$

We follow Chiarella and He (2001) and assume that the risk premium is constant over time. This assumption is made for the purpose of simplification.

Chartists, on the other hand, believe that the future return of the risky asset will be influenced by its past realizations and construct their forecasts such that the conditional mean is equal to the weighted sum of the constant risk premium and the past excess return:[4]

$$\mathrm{E}_{c,t}[\boldsymbol{R}_{t+1} - R_f] = d\delta + (1-d)(R_{t-1} - R_f), \tag{11}$$

where $d \in [0, 1]$ is the extrapolation parameter. We try to keep the model as simple and as tractable as possible and so we assume that chartists make use of only the previous period return realization when extrapolating the future return. This assumption can further be relaxed when the behavior of the simplest model is well-understood.

It is clear that the fundamentalists predictor (10) can be easily derived from (11) by setting the extrapolation parameter d equal to one.

The agents in our model cannot change their trading strategies once these were assigned to them. That is, we do not employ here evolutionary switching of beliefs (known as the adaptive belief system) designed by Brock and Hommes (1997) and Brock and Hommes (1998). This is so because the presence of wealth in the model precludes us from formulating it in terms of adaptive agents. If the agents were able to adopt another strategy, then every one of them would have her own wealth changing over time, thus, making the number of wealth trajectories as many as there are traders in the market. It is unknown to us how to keep track of this information when the number of agents is large, so we simplify our model and do not allow the agents to switch their beliefs keeping them fixed over time.

[4]We call this type chartists even though their beliefs are rather of fundamental nature because these contain the price fundamental value. We use this name in order to distinguish between the pure fundamental beliefs and somewhat sluggish (or trend chasing) fundamental expectations.

2.2. Heterogeneous reference points

In order to introduce the second source of heterogeneity in the model, we assume that there are two groups of agents with different reference levels of wealth that are called *ambitious* and *conservative*. The reference point in either group is defined as

$$m^k = m \pm \varepsilon, \tag{12}$$

where k denotes a particular group (ambitious or conservative), m stands for a common component of the reference level, and ε denotes a sufficiently small strictly positive deviation from m identical for both groups. The ambitious agents employ the augmented main component as the reference point ($m^a = m + \varepsilon$), while the conservative agents employ the diminished main component ($m^c = m - \varepsilon$). Notice that the smaller the magnitude of ε is, the more alike the fractions are. As well as for the beliefs, we assume that the ambitious and the conservative groups are fixed and the agents do not adopt alternative reference points.

In principle, unequal deviations ε should be considered for the two groups from the point of view of the realism of the model, however, this would introduce additional complications into the setup, which is already quite involved.

As a result, there are four fixed groups of agents whose sizes are exogenously determined: ambitious fundamentalists, conservative fundamentalists, ambitious chartists, and conservative chartists. Thus, along with the informational heterogeneity arising from different beliefs, there is also another source of heterogeneity emerging through different risk attitudes of the agents owing to their different assessment of their reference points. In fact, the agents with a higher reference point may be risk seeking in the region where the agents with a lower reference point remain risk averse. This may well lead to interactions and trade in the market among the agents as well as different beliefs do as shown in e.g. Brock and Hommes (1998). It is interesting to study how such reference points discrepancy affects the market dynamics.

Since all the four groups are fixed and the agents cannot leave the initially assigned to them group, the demands for the risky asset of all the agents within a group are identical. Besides, also the wealth is identical at every period of time for all the agents who reside in the same group given the same initial wealth.

Note that from now on subscripts in the formulas stand for particular belief types (e.g. $\mathrm{E}_j = \mathrm{E}_f$ for fundamentalists and $\mathrm{E}_j = \mathrm{E}_c$ for chartists), while superscripts denote certain reference points (e.g. $m^k = m^a$ for ambitious traders and $m^k = m^c$ for conservative ones) used by the agents.

The model appears to be quite complex and not only the demands and the wealth equations may change substantially when certain reference points are employed, but also the return equation should be defined in compliance with the way the reference level of wealth is chosen due to potential problems of the explicit formulation of the final system. Therefore, we leave the discussion of the price governing process for the next two sections as each of them is devoted to a particular reference point choice.

3. Current wealth as a reference point

In this section we construct a dynamical system and study how the reference point in the form of the current wealth of the agents affects asset return and wealth dynamics. This choice of the reference point seems quite natural to us as often humans tend to perceive their current resources as some reference level with respect to which they identify gains and losses.

In line with the discussion in the previous section, we model two groups of agents with respect to their reference point m^k: ambitious and conservative. We assume now that the reference level of wealth of the former is $m_t^a = W_t^a + \varepsilon$ and it is $m_t^c = W_t^c - \varepsilon$ of the later. Thus, the total reference point of the ambitious group of agents is slightly larger than their current wealth, while it is slightly smaller than the current wealth of the conservative agents.

We turn now to the return equation derivation. We follow Chiarella and He (2001) and make use of the Walrasian equilibrium design, so that the return and the wealth would be determined simultaneously at every period of time as in the real financial markets.

Denote $\Omega_{j,t}^k$ be the total wealth of the group (j,k) and $\pi_{j,t}^k$ be the common demand for all the agents within this group. The aggregate demand takes the following form:

$$\sum_{j,k} \pi_{j,t}^k \Omega_{j,t}^k = N p_t, \tag{13}$$

where N is the total number of shares of the risky asset, and p_t is its price at time t.

Let l_j^k be the number of individuals in the group (j,k), and denote n_j^k be the (fixed) proportion of agents contained in the (j,k)'s cluster relative to the total number of agents H in the market. It is clear that $\Omega_{j,t}^k = l_j^k \bar{W}_{j,t}^k = n_j^k H \bar{W}_{j,t}^k$, where $\bar{W}_{j,t}^k$ is the average wealth.

The equilibrium equation (13) can now be rewritten as

$$H \sum_{j,k} \pi_{j,t}^k \bar{W}_{j,t}^k n_j^k = N p_t. \tag{14}$$

Since the groups are fixed and the agents cannot switch between them, the average wealth in the group (j,k) is also the wealth of every agent in it: $\bar{W}_{j,t}^k = W_{j,t}^k$. It may also be considered as the wealth of the representative agent of a particular group.

We now replace t by $t-1$ in (14) to obtain the following:

$$\frac{\sum_{j,k} \pi_{j,t}^k W_{j,t}^k n_j^k}{\sum_{j,k} \pi_{j,t-1}^k W_{j,t-1}^k n_j^k} = \frac{p_t}{p_{t-1}} = R_t - \alpha, \tag{15}$$

where the right-hand side comes in from the definition of the gross risky asset return $R_t = (p_t + y_t)/p_{t-1}$ with α being the dividend yield y_t/p_{t-1}, which is equal to the sum of the risk premium and the risk-free asset rate of return: $\alpha = \delta + R_f$.

The final system can be obtained by collecting up (1) and (15):

$$\begin{cases} \boldsymbol{W}_{j,t+1}^k = W_{j,t}^k R_f + \pi_{j,t}^k W_{j,t}^k (\boldsymbol{R}_{t+1} - R_f), \\ \boldsymbol{R}_{t+1} = \delta + R_f + \frac{\sum_{j,k} \pi_{j,t+1}^k W_{j,t+1}^k n_j^k}{\sum_{j,k} \pi_{j,t}^k W_{j,t}^k n_j^k}, \end{cases} \tag{16}$$

where $\pi^k_{j,t}$ is given by (9).

Table 1 below summarizes the heterogeneity factors of the model.

	Fundamentalists	Chartists
Ambitious	$m^a_{f,t} = W^a_{f,t} + \varepsilon$	$m^a_{c,t} = W^a_{c,t} + \varepsilon$
	$\mathrm{E}^a_{f,t}[\boldsymbol{R}_{t+1} - R_f] = \delta$	$\mathrm{E}^a_{c,t}[\boldsymbol{R}_{t+1} - R_f] = d\delta + (1-d)R_{t-1}$
Conservative	$m^c_{f,t} = W^c_{f,t} - \varepsilon$	$m^c_{c,t} = W^c_{c,t} - \varepsilon$
	$\mathrm{E}^c_{f,t}[\boldsymbol{R}_{t+1} - R_f] = \delta$	$\mathrm{E}^c_{c,t}[\boldsymbol{R}_{t+1} - R_f] = d\delta + (1-d)R_{t-1}$

Table 1: Current wealth as a reference point: heterogeneity factors.

After plugging m^k_j and E^k_j into π^k_j in (16), the final system considerably simplifies and the return evolution equation becomes independent of wealth:

$$\boldsymbol{R}_{t+1} = \delta + R_f + \frac{\delta(n^c_f - n^a_f) + [d\delta + (1-d)(R_t - R_f)](n^c_c - n^a_c)}{\delta(n^c_f - n^a_f) + [d\delta + (1-d)(R_{t-1} - R_f)](n^c_c - n^a_c)}. \tag{17}$$

The form of (17) is somewhat surprising as one would expect explicit dependence of the return on the wealth of the agents. It is clear that the return also does not depend on the magnitude of the deviation ε, although the form of (17) is certainly affected by the fact that the reference point is constructed as a summation of the main component and the deviation for the ambitious agents and as a difference of the two for the conservative ones.

The return equation in (17) is a second order difference equation, and its local stability can be analyzed analytically. It is easy to find the unique steady state, which is given by $\delta + R_f + 1$.

For the purpose of tractability we will consider only pairs of groups. Any such pair can be elicited straightforwardly by putting to zero the agent proportions corresponding to the remaining two groups. For example, the return dynamics for the "ambitious fundamentalists - conservative chartists" pair can be derived from (17) by equating the proportions of conservative fundamentalists n^c_f and the proportions of ambitious chartists n^a_c to zero.

It becomes immediately clear that for the given reference point the return dynamics of the "ambitious fundamentalists - conservative fundamentalists" pair is degenerate as the right hand side of the return equation does not contain the state variable and the future return is equal to the steady state.

The local stability analysis of the system for the five remaining pairs can be implemented by analyzing the eigenvalues of the corresponding characteristic equations. However, the obtained analytic expressions of the eigenvalues are quite cumbersome to handle, so we make use of the numerical analysis and computer simulations instead to explore the impact of the parameters on the system's behavior. We are specifically interested in two parameters: the proportion of agents n_j^k in the market, and the extrapolation parameter d.

The simulations reveal that for all the four quasi-homogenous group pairs (those with either identical beliefs or with equal reference points) the steady state is globally stable. The only two pairs that generate instabilities in the market are "ambitious fundamentalists - conservative chartists" and "conservative fundamentalists - ambitious chartists".

In fact, the return time series for both groups, given certain values of the varying parameters, look quite complicated. They were generated for $d = 0.5$, $n_f^a = 0.6 = 1 - n_c^c$, $R_f = 1.1$, and $\delta = 0.5$, and are depicted in Figure 3.

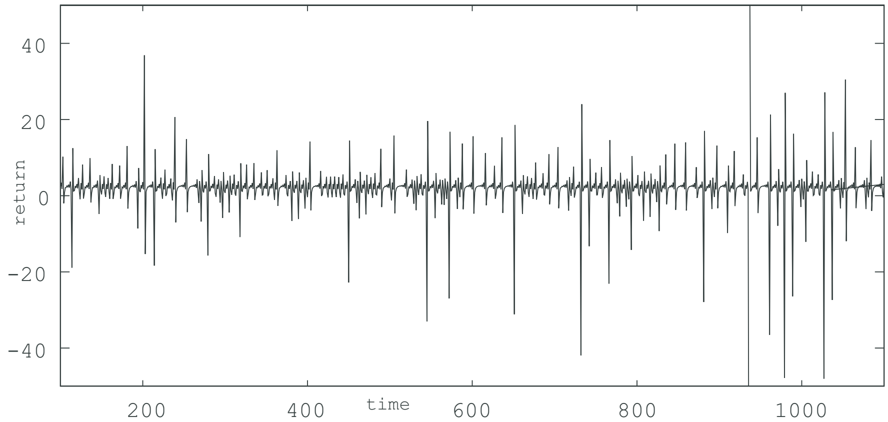


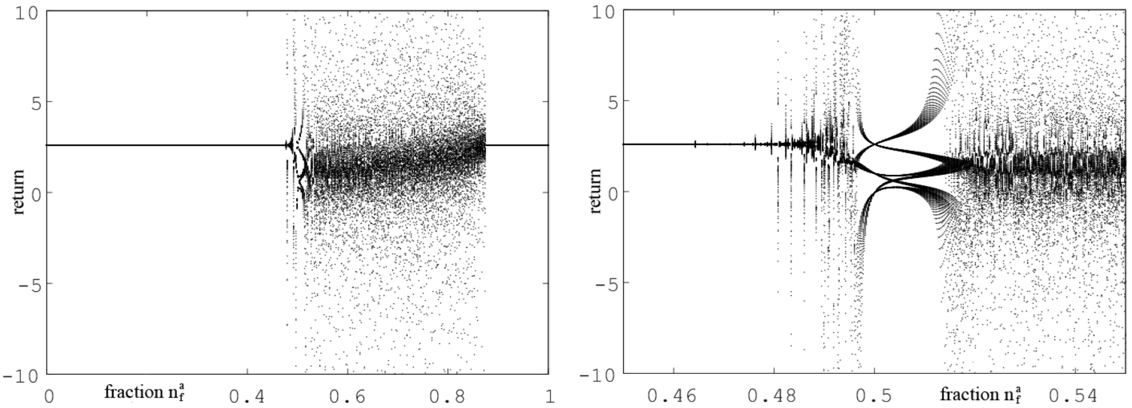
Figure 3: Return time series: "ambitious fundamentalists - conservative chartists".

We conduct the bifurcation analysis using numerical methods and explore the occurrence of chaos, periodic orbits and stable steady states using bifurcation diagrams. It turns out that the first bifurcation for the "ambitious fundamentalists - conservative chartists" pair occurs when there is a roughly equal number of both types of agents in the market $n_f^a = n_c^c = 0.5$ for any value of the extrapolation parameter d. Although, for $d = 1$ the system is stable, which is obvious because this value of the extrapolation parameter turns the extrapolation rule (11) into the fundamentalists' forecast (10).

Initial bifurcations quickly lead to chaotic behavior (Figure 4a), which soon change into periodic dynamics for the n_f^a slightly above 0.5 (Figure 4b). Afterwards, chaos appears again for quite a large interval of n_f^a until it dies down rapidly.

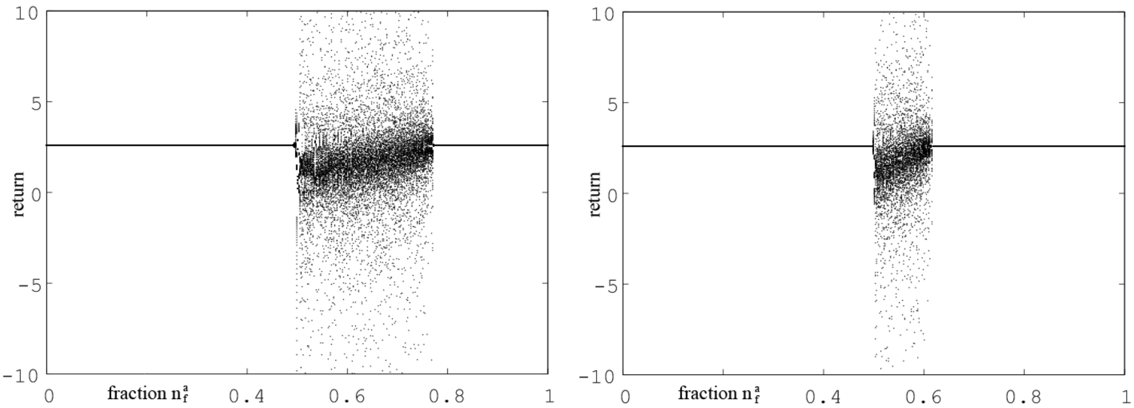
The ambitious fundamentalists proportion parameter value for which chaos disappears and the system's orbit converges to the single steady state varies with the extrapolation parameter. For the smallest value of $d = 0$ the stability region begins where n_f^a takes on the value of about 0.85, while for the highest tractable $d = 0.99$, the stability region begins nearly right after the first bifurcation occurs.

Consequently, the smaller the value of d is, the larger is the interval of the values of n_f^a for which the system exhibits chaotic behavior. This tendency is clearly seen if one compares Figure 4a (which is generated for $d = 0$), Figure 5a, and Figure 5b.



(a) Full bifurcation diagram (d=0). (b) Region of periodic behavior.

Figure 4: Bifurcation diagrams: "ambitious fundamentalists - conservative chartists".



(a) Extrapolation parameter d=0.6. (b) Extrapolation parameter d=0.9.

Figure 5: Chaotic decay with the increase of the extrapolation parameter.

The results of the analysis suggest that neither of the heterogeneity layers can trigger instabilities in the market in itself. There is a need for both the belief and the reference point sources of heterogeneity to be embedded in the model for it to invoke trading volume observed in the real financial markets. This conclusion, however, may depend on the reference point specification that was chosen, and in order to verify its robustness we introduce a different reference point in the following section.

4. Risk-free investment as a reference point

In this section we deal with another choice of the reference point, namely, with the wealth level corresponding to the investment in the risk-free asset in the current period $W_{j,t}^k R_f$. This reference point, just as the previous choice described in Section 3, represent an intuitively obvious reference level of wealth with any outcome above (or below) which one would identify as a gain (or a loss).

Analogously to the analysis in the previous section, we consider here two symmetric modifications of the reference point: the reference level is $W_t^a(R_f+\varepsilon)$ for ambitious agents, and it is $W_t^c(R_f-\varepsilon)$ for conservative ones. Table 2 below displays the heterogeneity factors regarding the second choice of the reference point.

Due to the particular demand form in (9) and its dependence on wealth, the final system derivation in explicit form using the Walrasian auctioneer protocol for the market clearing condition becomes extremely complicated. Therefore, we use a price adjustment rule that relates the change in the asset price and the aggregate demand of the investors:

$$\boldsymbol{p}_{t+1} - p_t = \sum_{j,k} z_{j,t}^k n_j^k, \tag{18}$$

where $z_{j,t}^k$ stands for the number of shares demanded by the agents ($z_{j,t}^k p_t = \pi_{j,t}^k W_{j,t}^k$).

	Fundamentalists	Chartists
Ambitious	$m_{f,t}^a = W_{f,t}^a(R_f+\varepsilon)$	$m_{c,t}^a = W_{c,t}^a(R_f+\varepsilon)$
	$\mathrm{E}_{f,t}^a[\boldsymbol{R}_{t+1} - R_f] = \delta$	$\mathrm{E}_{c,t}^a[\boldsymbol{R}_{t+1} - R_f] = d\delta + (1-d)R_{t-1}$
Conservative	$m_{f,t}^c = W_{f,t}^c(R_f-\varepsilon)$	$m_{c,t}^c = W_{c,t}^c(R_f-\varepsilon)$
	$\mathrm{E}_{f,t}^c[\boldsymbol{R}_{t+1} - R_f] = \delta$	$\mathrm{E}_{c,t}^c[\boldsymbol{R}_{t+1} - R_f] = d\delta + (1-d)R_{t-1}$

Table 2: Risk-free investment as a reference point: heterogeneity factors.

Analogous rule was used in Day and Huang (1990), Farmer and Joshi (2002) and Manzan and Westerhoff (2007) as a stylized way to represent a risk-neutral market-maker, a specific market institution, whose responsibility is similar to the Walrasian auctioneer: the market-maker aggregates the demands and adjusts the future price of the asset. In such a way the price adjustment rule describes the relation between the number of shares demanded and supplied and the price change due to this net order imbalance.

In general, the price adjustment function $f(p,z)$ contains a reaction coefficient of the market-maker, which we set equal unity for simplification.

In order to obtain return instead of price dynamics, we divide the price adjustment rule (18) by its lagged version in the same way as in (15). As before, we assume that the dividend yield $\alpha = y_t/p_{t-1}$ is constant and equal to the sum of the risk premium δ and the risk-free rate of return R_f.

With the price adjustment rule the final system looks as follows:

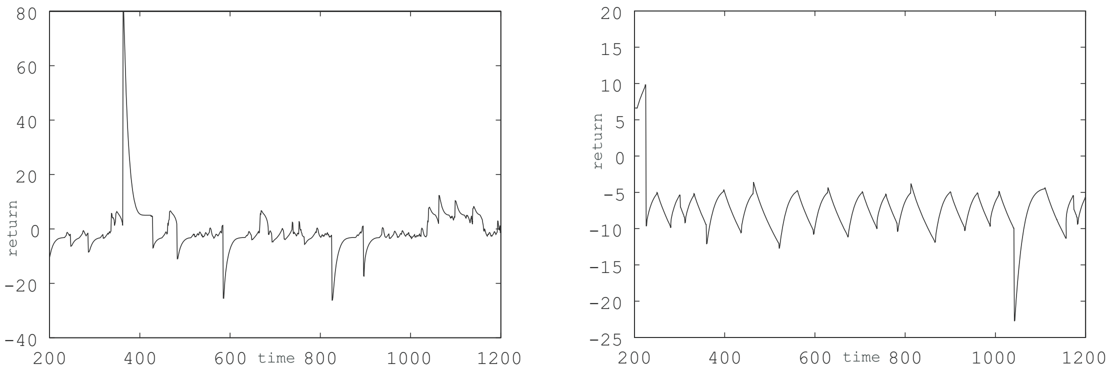
$$\begin{cases} \boldsymbol{W}^k_{j,t+1} = W^k_{j,t}R_f + \pi^k_{j,t}W^k_{j,t}(\boldsymbol{R}_{t+1} - R_f), \\ \\ \boldsymbol{R}_{t+1} = R_t + \frac{1}{R_t - \delta - R_f}\frac{\sum_{j,k}\pi^k_{j,t}W^k_{j,t}n^k_j}{\sum_{j,k}\pi^k_{j,t-1}W^k_{j,t-1}n^k_j}, \end{cases} \tag{19}$$

where $\pi^k_{j,t}$ given by (9).

Note that the wealth process in (19) is intrinsically growing. In order to make the system analytically tractable the expansion of the system should be removed (as e.g. in Chiarella and He (2001) and Anufriev et al. (2006)). However, the dependence on wealth in the demand function (9) makes this task very difficult to implement. In fact, we were unable to rescale the system and remove exogenous expansion and rely on computer simulations for its further analysis.

Because the wealth is growing with time, the computer simulations cannot capture the return dynamics when the wealth value becomes too large to handle numerically. Therefore, we can observe only truncated time series of the return. Nevertheless, this allows us to verify whether the conclusion of the previous section is robust with respect to the reference point choice or not. That is, from the time series we are able to see whether there is any trading volume when there are only quasi-homogenous pairs of groups in the market.

In fact, all the six pairs generate irregular time series of the asset return. Figure 6a exhibits an example of the time series for the "ambitious chartists - conservative chartists" pair, while Figure 6b depicts the return dynamics for the "conservative fundamentalists - conservative chartists" pair of groups.



(a) "ambitious chartists - conservative chartists".

(b) "conservative fundamentalists - conservative chartists".

Figure 6: Quasi-homogenous pairs return time series.

Two straightforward implications of the results of the simulations are the following. Firstly, reference point heterogeneity alone may indeed generate trading volume in the market, even if the beliefs of the agents are kept identical. Secondly, a reference point choice not only strongly affects the behavior of the model, but may also influence the relative impact of the two heterogeneity layers on the market dynamics.

5. Concluding remarks

In this paper we introduced the second source of heterogeneity into a modified framework of Brock and Hommes (1998) and Chiarella and He (2001). Apart from the heterogeneity of beliefs, which has been extensively studied in numerous multi-agent models, we explored how heterogeneity of reference points of the market traders affects asset return and wealth dynamics.

We used the notion of an S-shaped utility presented in Gerasymchuk (2007) to construct the demand for the risky asset. In order to sustain realistic underlying risk aversion properties of the demand function, we modified the demand corresponding to the normal utility by replacing the risk aversion coefficient by the one of the arctangent utility.

Two distinct reference points of the agents were considered, whose choice was quite natural: the agent's current wealth and the risk-free investment. We studied both reference point choices and their impact on the resulting market dynamics by means of numerical analysis and computer simulations.

The simulation results revealed that the heterogeneity of reference points can lead to market instabilities and persistent trading volume even if the beliefs of market traders are kept homogeneous.

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